

Calculus III: Optional Coursework 1
M.A.H. MacCallum 2007

Note: Although this course work is optional, those who decide not to do it would be well-advised to make sure they understand the solutions, since related questions may appear in the test and the final exam.

The deadline is 4PM on Tuesday Oct 30th. Hand-in is to the green box on the ground floor of the Maths building.

1. Using the Divergence Theorem, evaluate $\int_{\mathcal{S}} \mathbf{A} \cdot \mathbf{n} \, dS$ where $\mathbf{A} = (y+z)\mathbf{i} + (x+y)\mathbf{j} + (x+z)\mathbf{k}$ and $\mathcal{S}$ is the surface of the region $0 \leq y \leq 2$, $0 \leq x^2 + z^2 \leq 1$. [If feeling strong, you might like to try doing this by direct integration over the surface, to see how much more awkward it is!] [16]

Answer

The divergence of $\mathbf{A}$ is

$$\frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(x+y) + \frac{\partial}{\partial z}(x+z) = 2$$

and the region is a cylinder with circular base of radius 1 in the (x, z) plane and length 2. Its volume is therefore $\pi \cdot 1^2 \cdot 2 = 2\pi$. So the answer is 4π .

Marks: 4 for correct use of Div Thm, 4 for correct divergence, 4 for identification of volume (in a sketch or words), 4 for final result.

2. An important theorem in complex variables (called Cauchy's theorem) states that if there are two well-behaved functions $u(x, y)$ and $v(x, y)$ satisfying [12]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and an arbitrary closed curve C in the x, y plane then

$$\oint_C (u + iv) (dx + i dy) = 0.$$

Show that Cauchy's theorem is a special case of Green's theorem in the plane.

Hint: Try separating out the real and imaginary parts of the integral.

Answer

The real and imaginary parts of the integral are given by

$$I \equiv \int (u \, dx - v \, dy) + i \int (v \, dx + u \, dy) .$$

Applying Green's theorem gives

$$I = - \iint \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx \, dy + i \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx \, dy .$$

The given relation between the derivatives makes both integrands zero.

Marks: Separating real and imaginary parts (4), applying Green's theorem (4), simplifying (4).

Comment: In complex analysis one would write $f(z) = u + iv$ where $z = x + iy$ and the result is $\int_C f dz = 0$.

3. Evaluate both sides of Stokes's Theorem for $\mathbf{F} = xy\mathbf{j} + yz\mathbf{k}$, for a closed path in the $x = 0$ plane consisting of straight path segments as follows: from the origin to $(0, a, 0)$; from there to $(0, a, b)$; from there to $(0, 0, b)$; and back to the origin, and for a suitably chosen surface. Show that the two sides are equal. [20]

Answer

$\nabla \times \mathbf{F} = z\mathbf{i} + y\mathbf{k}$ and the surface integral can be taken over a rectangle S in the (y, z) plane with $\mathbf{n} = \mathbf{i}$. Its value is thus

$$\begin{aligned} \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \int_0^b \int_0^a (z\mathbf{i} + y\mathbf{k}) \cdot \mathbf{i} dy dz \\ &= \int_0^b \int_0^a z dy dz = \frac{1}{2}ab^2. \end{aligned}$$

The integral along the curve has four parts.

1. $\mathbf{r} = t\mathbf{j}$, $0 \leq t \leq a$. Here $\mathbf{F} \cdot d\mathbf{r} = 0$.
2. $\mathbf{r} = a\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq b$. Here $\mathbf{F} \cdot d\mathbf{r} = at$ and the integral is $\int_0^b at dt = ab^2/2$.
3. $\mathbf{r} = (a-t)\mathbf{j} + b\mathbf{k}$, $0 \leq t \leq a$. Here $d\mathbf{r} = -\mathbf{j}$ and $\mathbf{F} \cdot d\mathbf{r} = 0$.
4. $\mathbf{r} = (b-t)\mathbf{k}$, $0 \leq t \leq b$. Yet again $\mathbf{F} \cdot d\mathbf{r} = 0$.

So the total agrees with the first evaluation.

Marks: 4 for the curl, 6 for the surface integration, 3 for a correct parametrization of at least one part of the curve and 1 more for 3rd or 4th part correctly parametrized, 1 each for the zero contributions, 3 for the non-zero one.

4. Using Stokes's Theorem, evaluate $\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} dS$ for $\mathbf{A} = -y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$ and the surface S given by $z \geq 0$, $x^2 + y^2 + z^2 = a^2$. [16]

Answer

Using Stokes's theorem $\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} dS = \int_C \mathbf{A} \cdot d\mathbf{r}$. The curve C bounding S is the circle $x^2 + y^2 = a^2$. On it $\mathbf{A} \cdot d\mathbf{r} = -y dx$. Round it we can use a parameter θ such that $x = a \cos \theta$, $y = a \sin \theta$ (polar coordinates, in fact).

The line integral is thus

$$\int_C -y dx = \int_0^{2\pi} (-a \sin \theta)(-a \sin \theta) d\theta$$

which easily gives πa^2 .

Marks: correct use of Stokes's theorem 4, parametrizing C in a correct way (there are many) 4, correct integral formed 4, final answer 4.

Note: a less likely but also correct argument would be to note that the answer must, by Stokes's theorem, be the same for the flat disc in the (x, y) plane, $\nabla \times \mathbf{A} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $d\mathbf{S} = \mathbf{k}dx dy$ so we have $\int \int dx dy$ over a circle which is the area of the circle and gives the same result. Divide the marks in a similar manner.

5. Show that the following are all zero: [36]

- (a) $\oint \mathbf{r} \cdot d\mathbf{r}$ over an arbitrary closed path;
- (b) $\oint d\mathbf{S}$ over an arbitrary closed surface [*Hint:* Try taking the $\mathbf{k}$ component.];
- (c) $\oint \mathbf{r} \times d\mathbf{S}$ over an arbitrary closed surface;

where in each case the arbitrary curve or surface may be assumed to be sufficiently smooth for the applicability of any theorems you use.

Answer

- (a) Using Stokes theorem

$$\oint \mathbf{r} \cdot d\mathbf{r} = \int (\nabla \times \mathbf{r}) \cdot d\mathbf{S} = 0$$

- (b) Consider the $\mathbf{k}$ component:

$$\mathbf{k} \cdot \oint d\mathbf{S} = \int (\nabla \cdot \mathbf{k}) dV = 0$$

by the divergence theorem. This applies similarly for the $\mathbf{i}$ and $\mathbf{j}$ components so the original vector is zero.

- (c) Again consider the $\mathbf{k}$ component:

$$\mathbf{k} \cdot \oint \mathbf{r} \times d\mathbf{S} = \oint (\mathbf{k} \times \mathbf{r}) \cdot d\mathbf{S} = \int \nabla \cdot (\mathbf{k} \times \mathbf{r}) dV ,$$

again by the divergence theorem. The divergence evaluates to zero, hence the integral is zero. Complete by the same argument as the previous part.

Marks: (a) Invoking Stokes' theorem correctly 4, realizing and using that $\nabla \times \mathbf{r} = 0$, 4; (b) taking component correctly 2, rewriting triple product 4, using divergence theorem 4, showing divergence is zero 4, final step 4; (c) again using components 6 (4+2 for last step), using divergence theorem 4.